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Maximal and Primitive Elements in Weyl Modules for Type A_2

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The submodule structure of Weyl modules for SL_3 has been determined in [DS, I, K] and by Cline (unpublished). The main technique used in [DS] is an analysis of the intertwining homomorphisms between certain Weyl modules. The approach in [I, K] is essentially a direct calculation. Also, Jantzen's translation principle plays an important role in these works. For a few special cases, Irving gives some maximal and primitive elements in Verma modules of the Frobenius kernel and Weyl modules (see [I]). In this paper we will find out all maximal and primitive elements in Verma modules of the quantized Frobenius kernel and Weyl modules for type A_2; see Sections 2–3. Thus we understand the structure of the Verma modules and the Weyl modules more explicitly. Our method is based on the main results in [X1] and some ideas in [I] and is rather simple. In this paper we only work with quantized enveloping algebras at roots of 1 (Lusztig version); for hyperalgebras the approach is completely similar. The method also works for type B_2 (see [X2]). The author also did some computations for type A_3 using the method; although the size of the computation is large, it still does not seem difficult to work out the structure of Weyl modules for type A_3.

1. MAXIMAL AND PRIMITIVE ELEMENTS

In this section we fix notation and give the definition for maximal and primitive elements. We refer to [L1–L4, X1] for additional information.

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1.1. Let U_ξ be a quantized enveloping algebra (over $\mathbf{Q}(\xi)$) at a root ξ of 1. We assume that the rank of the associated Cartan matrix is n and the order of $\xi \geq 3$. As usual, the generators of U_ξ are denoted by $E_i^{(a)}, F_i^{(a)}, K_i, K_i^{-1}$, etc. Let $\mathbf{u}$ be the Frobenius kernel and $\tilde{\mathbf{u}}$ the subalgebra of U_ξ generated by all elements in $\mathbf{u}$ and in the zero part of U_ξ. For $\lambda \in \mathbf{Z}^n$ and a U_ξ-module (or $\tilde{\mathbf{u}}$-module M) we denote by M_λ the λ-weight space of M. A nonzero element in M_λ will be called a vector of weight λ or a weight vector. Let m be a weight vector of a U_ξ-module (resp. $\tilde{\mathbf{u}}$-module) M. We call m maximal if $E_i^{(a)}m = 0$ for all i and $a \geq 1$ (resp. $E_\alpha m = 0$ for all root vectors E_α in the positive part of $\tilde{\mathbf{u}}$). We call m a primitive element if there exist two submodules $M_2 \subset M_1$ of M such that $m \in M_1$ and the image in M_1/M_2 of m is maximal. Obviously, maximal elements are primitive. We have:

(a) *Let $m \in M$ be a weight vector and let P_1 be the submodule of M generated by m. Then m is primitive if and only if the image in P_1/P_2 of m is maximal for some submodule P_2 of P_1.*

Proof. Assume m is primitive. Then we can find submodules $M_2 \subset M_1$ of M such that $m \in M_1$ and the image in M_1/M_2 of m is maximal. Let $P_2 = P_1 \cap M_2$. Then the image in P_2/P_2 of m is maximal. The other direction is obvious.

(b) *If m is a primitive element of weight λ, then $L(\lambda)$ (or $\tilde{L}(\lambda)$) is a composition factor of M (depending on whether M is a U_ξ-module or a $\tilde{\mathbf{u}}$-module). Here and later we write $L(\lambda)$ (resp. $\tilde{L}(\lambda)$) for an irreducible U_ξ-module (resp. $\tilde{\mathbf{u}}$-module) of highest weight λ.*

Proof. Let P_1, P_2 be as in (a) and let Q_1 be a maximal submodule of P_1 which includes P_2. Then $N = P_1/Q_1$ is irreducible and the image in N of m is maximal. Therefore N is necessarily isomorphic to $L(\lambda)$ (or $\tilde{L}(\lambda)$).

(c) *Let M and N be modules and $\phi: M \to N$ a homomorphism. Let m be a weight vector in M. If $\phi(m)$ is a primitive element of N, then m is a primitive element of M.*

Proof. Let P_1 and Q_1 be the submodules of M and N generated by m and $\phi(m)$, respectively. Then ϕ induces a surjective homomorphism $\phi_1 : P_1 \to Q_1$. If $\phi(m) = \phi_1(m)$ is primitive, by (a), we can find a submodule Q_2 of Q_1 such that the image in Q_1/Q_2 of $\phi_1(m)$ is maximal. Let $P_2 = \phi_1^{-1}(Q_1)$. Then ϕ_1 induces an isomorphism $\overline{\phi}_1 : P_1/P_2 \to Q_1/Q_2$. Therefore the image in P_1/P_2 of m is maximal. In particular, m is primitive.

(d) *Let M, N, ϕ, m be as in* (c) *and assume $\phi(m) \neq 0$. If m is a primitive element of M, then either $\phi(m)$ is a primitive element of N or $\phi(P_1) = \phi(P_2)$, where P_1 is the submodule of M generated by m and P_2 is any submodule of P_1 such that the image in P_1/P_2 of m is maximal.*

Proof. If $\phi(P_1) \neq \phi(P_2)$, then $Q = \phi(P_1)/\phi(P_2) \neq 0$. In this case $\phi(m)$ is in $\phi(P_1)$ but not in $\phi(P_2)$, and the image in Q of $\phi(m)$ is maximal, hence $\phi(m)$ is primitive.

If $\phi(P_1) = \phi(P_2)$ for any submodule P_2 of P_1 such that the image in P_1/P_2 of m is maximal, then $\phi(m)$ is not primitive. Otherwise, let P_2 be as in the proof of (c). Then the image in P_1/P_2 of m is maximal and $\phi(P_1) \neq \phi(P_2)$; this contradicts the assumption.

(e) *Let M, N, ϕ, m be as in* (c) *and assume $\phi(m) \neq 0$. If m is a maximal element of M, then $\phi(m)$ is a maximal element of N.*

Proof. This is obvious.

We shall denote by $\tilde{Z}(\lambda)$ the Verma module of $\tilde{\mathbf{u}}$ with highest weight λ and denote by $\tilde{1}_\lambda$ a nonzero element in $\tilde{Z}(\lambda)_\lambda$. Recall that to define U_ξ we need to choose $d_i \in \{1, 2, 3\}$ such that $(d_i a_{ij})$ is symmetric, where (a_{ij}) is the concerned $n \times n$ Cartan matrix. Let l_i be the order of ξ^{2d_i}. For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbf{Z}^n$ we set $\mathbf{l}\lambda = (l_1\lambda_1, \ldots, l_n\lambda_n)$. We call λ $\mathbf{l}$-restricted if $0 \leq \lambda_i \leq l_i - 1$ for all i. Denote by $\mathbf{Z}^n_{+,\mathbf{l}}$ the set of all $\mathbf{l}$-restricted elements in $\mathbf{Z}^n$. The following fact is well known.

(f) *Let $\lambda' \in \mathbf{Z}^n_{+,\mathbf{l}}$, $\lambda'' \in \mathbf{Z}^n$. Set $\lambda = \lambda' + \mathbf{l}\lambda'' \in \mathbf{Z}^n$. Then $F_i^{(\lambda'_i+1)}\tilde{1}_\lambda$ is maximal if $\lambda'_i \neq l_i - 1$.*

2. STRUCTURE OF $\tilde{Z}(\lambda)$ FOR TYPE A_2

2.1. From now on we assume that U_ξ is of type A_2. In this section we determine the maximal and primitive elements in $\tilde{Z}(\lambda)$ (or equivalently in any highest weight module of $\tilde{\mathbf{u}}$) and the submodule structure of $\tilde{Z}(\lambda)$, see Theorems 2.2–2.7. For completeness, we give the definition of U_ξ and $\tilde{Z}(\lambda)$.

Let U be the associative algebra $\mathbf{Q}(v)$ (v an indeterminate) generated by E_i, F_i, K_i, K_i^{-1} ($i = 1, 2$) with relations

$$K_1K_2 = K_2K_1, \qquad K_iK_i^{-1} = K_i^{-1}K_i = 1$$

$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{v - v^{-1}}$$

$$E_i^2E_j - (v + v^{-1})E_iE_jE_i + E_jE_i^2 = 0 \qquad \text{if } i, j \text{ are different}$$

$$F_i^2F_j - (v + v^{-1})F_iF_jF_i + F_jF_i^2 = 0 \qquad \text{if } i, j \text{ are different}.$$

Let U' be the $A = Z[v, v^{-1}]$-subalgebra of U generated by all $E_i^{(a)} = E^a/[a]!$, $F_i^{(a)} = F^a/[a]!$, K_i, K_i^{-1}, $a \in \mathbf{N}$, $i = 1, 2$, where $[a]! =$

$\prod_{h=1}^{a}((v^h - v^{-h})/(v - v^{-1}))$ if $a \geq 1$ and $[0]! = 1$. Note that

$$\begin{bmatrix} K_i, c \\ a \end{bmatrix} = \prod_{h=1}^{a} \frac{v^{c-h+1}K_i - v^{-c+h-1}K_i^{-1}}{v^h - v^{-h}}$$

is in U' for all $c \in \mathbf{Z}$, $a \in \mathbf{N}$. We understand that $\left[\begin{smallmatrix} K_i, c \\ a \end{smallmatrix}\right] = 1$ if $a = 0$. Also $F_{12}'^{(a)} = (F_1F_2 - vF_2F_1)^a/[a]!$ and $F_{12}^{(a)} = (F_2F_1 - vF_1F_2)^a/[a]!$ are in U' for all $a \in \mathbf{N}$. Regard $\mathbf{Q}(\xi)$ as an A-algebra by specializing v to ξ. Then $U_\xi = U' \otimes_A \mathbf{Q}(\xi)$.

For convenience, the images in U_ξ of $E_i^{(a)}, F_i^{(a)}, F_{12}'^{(a)}, F_{12}^{(a)}, K_i, K_i^{-1}, \left[\begin{smallmatrix} K_i, c \\ a \end{smallmatrix}\right]$, etc., will be denoted by the same notation, respectively. Let $l' = l_1 = l_2$. In U_ξ we have $E_i^{l'} = F_i^{l'} = 0$. The Frobenius kernel $\mathbf{u}$ of U_ξ is the subalgebra of U_ξ generated by all $E_i, F_i, K_i, K_i^{-1}, i = 1, 2$. Its negative part $\mathbf{u}^-$ is generated by all F_i. Note that $F_{12}'^{(a)}, F_{12}^{(a)}$ are in $\mathbf{u}^-$ if $0 \leq a \leq l' - 1$. The subalgebra $\tilde{\mathbf{u}}$ of U_ξ is generated by all $E_i, F_i, K_i, K_i^{-1}, \left[\begin{smallmatrix} K_i, c \\ a \end{smallmatrix}\right]$, $i = 1, 2$; $c \in \mathbf{Z}$, $a \in \mathbf{N}$.

For $\lambda = (\lambda_1, \lambda_2) \in \mathbf{Z}^2$, we denote by $\tilde{I}_\lambda$ the left ideal of $\tilde{\mathbf{u}}$ generated all $E_i, K_i - \xi^{\lambda_i}, \left[\begin{smallmatrix} K_i, c \\ a \end{smallmatrix}\right] - \left[\begin{smallmatrix} \lambda_i + c \\ a \end{smallmatrix}\right]_\xi$. (We denote by $\left[\begin{smallmatrix} b \\ a \end{smallmatrix}\right]_\xi$ the value at ξ of $\prod_{h=1}^{a}((v^{b-h+1} - v^{-b+h-1}/(v^h - v^{-h}))$ for any $b \in \mathbf{Z}$ and $a \in \mathbf{N}$.) The Verma module $\tilde{Z}(\lambda)$ of $\tilde{\mathbf{u}}$ is defined to be $\tilde{\mathbf{u}}/\tilde{I}_\lambda$. Recall that we denote by $\tilde{1}_\lambda$ a nonzero vector in $\tilde{Z}(\lambda)_\lambda$. The following result is a special case of [X1, 4.2 (ii)].

(a) Assume $0 \leq a, b \leq l' - 1, c, d, \in \mathbf{Z}$, and let $\mu = (l'c - 1 + a, l'd - 1 + b)$. Then the element $F_2^{(a)}F_1^{(a+b)}F_2^{(b)}\tilde{1}_\mu = F_1^{(b)}F_2^{(a+b)}F_1^{(a)}\tilde{1}_\mu$ is maximal in $\tilde{Z}(\mu)$ and generates the unique irreducible submodule of $\tilde{Z}(\mu)$. The irreducible submodule is isomorphic to $\tilde{L}(l'c - 1 - b, l'd - 1 - a)$.

We shall need a few formulas, which are due to Lusztig.

(b) In U_ξ we have

$$F_i^{(a)}F_i^{(b)} = \begin{bmatrix} a + b \\ a \end{bmatrix}_\xi F_i^{(a+b)}$$

$$F_1^{(a)}F_2^{(b)} = \sum_{0 \leq r \leq a, b} \xi^{(a-r)(b-r)}F_2^{(b-r)}F_{12}'F_1^{(a-r)}$$

$$F_2^{(a)}F_1^{(b)} = \sum_{0 \leq r \leq a, b} \xi^{(a-r)(b-r)}F_1^{(b-r)}F_{12}^{(r)}F_2^{(a-r)}.$$

(c) Assume $0 \le a_0, b_0 \le l' - 1$, and $a_1, b_1 \in \mathbf{Z}$. We have

$$\begin{bmatrix} a_0 + a_1 l' \\ b_0 + b_1 l' \end{bmatrix}_\xi = \xi^{(a_0 b_1 - a_1 b_0) l' + (a_1+1) b_1 l'^2} \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}_\xi \begin{pmatrix} a_1 \\ b_1 \end{pmatrix},$$

where $\binom{a_1}{b_1}$ is the ordinary binomial coefficient.

Let $\alpha_1 = (2, -1)$, $\alpha_2 = (-1, 2) \in \mathbf{Z}^2$. The set of positive roots is $R^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$. Let W be the Weyl group generated by the simple reflections s_i corresponding to α_i. Let $\lambda = (a, b) \in \mathbf{Z}^2_{+,\mathbf{l}}$ such that $\langle \lambda + \rho, \beta^\vee \rangle \ne l'$ for all $\beta \in R^+$. We consider the W-orbit of λ (dot action), which consists of the following 6 elements,

$$\lambda,\ s_1.\lambda = \lambda - (a+1)\alpha_1, \qquad s_2.\lambda = \lambda - (b+1)\alpha_2,$$
$$s_2 s_1.\lambda = \lambda - (a+1)\alpha_1 - (a+b+2)\alpha_2,$$
$$s_1 s_2.\lambda = \lambda - (a+b+2)\alpha_1 - (b+1)\alpha_2,$$
$$s_1 s_2 s_1.\lambda = \lambda - (a+b+2)\alpha_1 - (a+b+2)\alpha_2.$$

THEOREM 2.2. *Let $\lambda' = (a, b) \in \mathbf{Z}^2_{+,\mathbf{l}}$, $\lambda'' \in \mathbf{Z}^2$. Set $\lambda = \lambda' + \mathbf{l}\lambda'' \in \mathbf{Z}^2$. Assume that $a + b + 2 < l'$. Then*

(i) *The following 6 elements are maximal in $\tilde{Z}(\lambda)$:*

$$\tilde{1}_\lambda, \qquad F_1^{(a+1)}\tilde{1}_\lambda, \qquad F_2^{(b+1)}\tilde{1}_\lambda, \qquad F_2^{(a+b+2)}F_1^{(a+1)}\tilde{1}_\lambda,$$
$$F_1^{(a+b+2)}F_2^{(b+1)}\tilde{1}_\lambda, \qquad F_2^{(a+1)}F_1^{(a+b+2)}F_2^{(b+1)}\tilde{1}_\lambda.$$

(ii) *The following 3 elements are primitive elements in $\tilde{Z}(\lambda)$ but not maximal*:

$$\left(F_1^{(a+b+2)}F_2^{(l')} - \xi^{l'(a+b+2)}F_2^{(l')}F_1^{(a+b+2)}\right)F_2^{(b+1)}\tilde{1}_\lambda,$$
$$\left(F_2^{(a+b+2)}F_1^{(l')} - \xi^{l'(a+b+2)}F_1^{(l')}F_2^{(a+b+2)}\right)F_1^{(a+1)}\tilde{1}_\lambda,$$
$$F_1^{(l'-a-1)}F_2^{(l')}F_1^{(a+1)}\tilde{1}_\lambda.$$

(*We also can choose the third one to be $F_2^{(l'-b-1)}F_1^{(l')}F_2^{(b+1)}\tilde{1}_\lambda$.*) *Moreover there are no maximal elements in $\tilde{Z}(\lambda)$ which have the same weight with any of the above three elements.*

(iii) *The maximal and primitive elements in* (i)–(ii) *provide nine composition factors of* $\tilde{Z}(\lambda)$, *which are*

$$\tilde{L}(\lambda), \qquad \tilde{L}(\lambda - (a+1)\alpha_1), \qquad \tilde{L}(\lambda - (b+1)\alpha_2),$$

$$\tilde{L}(\lambda - (a+1)\alpha_1 - (a+b+2)\alpha_2),$$

$$\tilde{L}(\lambda - (a+b+2)\alpha_1 - (b+1)\alpha_2),$$

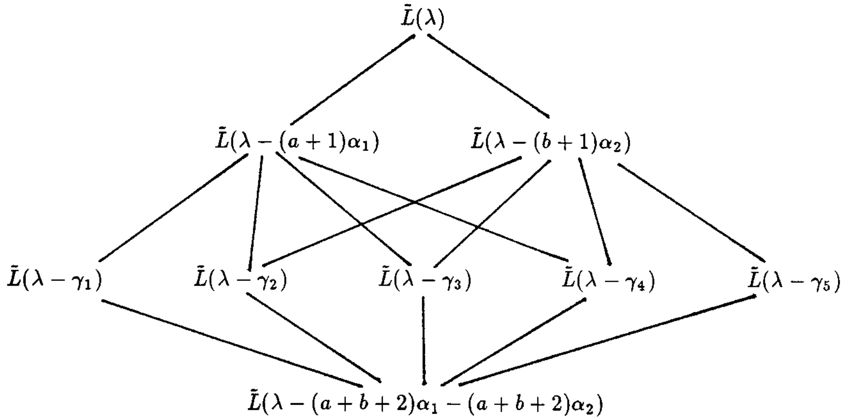
$$\tilde{L}(\lambda - (a+b+2)\alpha_1 - (a+b+2)\alpha_2),$$

$$\tilde{L}(\lambda - (a+b+2)\alpha_1 - (l'+b+1)\alpha_2),$$

$$\tilde{L}(\lambda - (l'+a+1)\alpha_1 - (a+b+2)\alpha_2), \qquad \tilde{L}(\lambda - l'\alpha_1 - l'\alpha_2).$$

Moreover, $\tilde{Z}(\lambda)$ *has only the nine composition factors.*

(iv) *The submodule lattice of* $\tilde{Z}(\lambda)$ *is* (*see* [DS, I, K])



where $\gamma_1 = (l'+a+1)\alpha_1 + (a+b+2)\alpha_2$, $\gamma_2 = (a+1)\alpha_1 + (a+b+2)\alpha_2$, $\gamma_3 = l'\alpha_1 + l'\alpha_2$, $\gamma_4 = (a+b+2)\alpha_1 + (b+1)\alpha_2$, $\gamma_5 = (a+b+2)\alpha_1 + (l'+b+1)\alpha_2$.

Proof. According to Subsection 1.1(e)–(f), we see (i) is true.

Using Subsections 1.1(e)–(f) and 2.1(a) we get

(1) Let $\mu = \lambda + (l'-a-1)\alpha_1$. The following elements are maximal in $\tilde{Z}(\mu)$:

$$\tilde{1}_\mu, \qquad F_1^{(l'-a-1)}\tilde{1}_\mu, \qquad F_2^{(a+b+2)}\tilde{1}_\mu, \qquad F_2^{(b+1)}F_1^{(l'-a-1)}\tilde{1}_\mu,$$

$$F_1^{(b+1)}F_2^{(a+b+2)}\tilde{1}_\mu, \qquad F_2^{(l'-a-1)}F_1^{(l'+b+1)}F_2^{(a+b+2)}\tilde{1}_\mu.$$

(2) Let $\mu = \lambda + (l' - b - 1)\alpha_2$. The following elements are maximal in $\tilde{Z}(\mu)$:

$$\tilde{1}_\mu, \quad F_1^{(a+b+2)}\tilde{1}_\mu, \quad F_2^{(l'-b-1)}\tilde{1}_\mu, \quad F_2^{(a+1)}F_1^{(a+b+2)}\tilde{1}_\mu,$$
$$F_1^{(b+1)}F_2^{(l'-b-1)}\tilde{1}_\mu, \quad F_1^{(l'-b-1)}F_2^{(l'+a+1)}F_1^{(a+b+2)}\tilde{1}_\mu.$$

(3) Let $\mu = \lambda + (l' - a - 1)\alpha_1 + (l' - a - b - 2)\alpha_2$. The following elements are maximal in $\tilde{Z}(\mu)$:

$$\tilde{1}_\mu, \quad F_1^{(b+1)}\tilde{1}_\mu, \quad F_2^{(l'-a-b-2)}\tilde{1}_\mu, \quad F_2^{(l'-a-1)}F_1^{(b+1)}\tilde{1}_\mu,$$
$$F_1^{(l'-a-1)}F_2^{(l'-a-b-2)}\tilde{1}_\mu, \quad F_2^{(b+1)}F_1^{(l'-a-1)}F_2^{(l'-a-b-2)}\tilde{1}_\mu.$$

(4) Let $\mu = \lambda + (l' - a - b - 2)\alpha_1 + (l' - b - 1)\alpha_2$. The following elements are maximal in $\tilde{Z}(\mu)$:

$$\tilde{1}_\mu, \quad F_1^{(l'-a-b-2)}\tilde{1}_\mu, \quad F_2^{(a+1)}\tilde{1}_\mu, \quad F_2^{(l'-b-1)}F_1^{(l'-b-2)}\tilde{1}_\mu,$$
$$F_1^{(l'-b-1)}F_2^{(a+1)}\tilde{1}_\mu, \quad F_1^{(a+1)}F_2^{(l'-b-1)}F_1^{(l'-a-b-2)}\tilde{1}_\mu.$$

(5) Let $\mu = \lambda + (l' - a - b - 2)\alpha_1 + (l' - a - b - 2)\alpha_2$. The following elements are maximal in $\tilde{Z}(\mu)$:

$$\tilde{1}_\mu, \quad F_1^{(l'-b-1)}\tilde{1}_\mu, \quad F_2^{(l'-a-1)}\tilde{1}_\mu, \quad F_2^{(l'-a-b-2)}F_1^{(l'-b-1)}\tilde{1}_\mu,$$
$$F_1^{(l'-a-b-2)}F_2^{(l'-a-1)}\tilde{1}_\mu, \quad F_1^{(l'-a-1)}F_2^{(2l'-a-b-2)}F_1^{(l'-b-1)}\tilde{1}_\mu.$$

Now we consider the homomorphism,

$$\tilde{Z}(\lambda) \to \tilde{Z}(\lambda + (l' - a - 1)\alpha_1), \qquad \tilde{1}_\lambda \to s = F_1^{(l'-a-1)}\tilde{1}_{\lambda + (l'-a-1)\alpha_1}.$$

Let

$$x = \left(F_1^{(a+b+2)}F_2^{(l')} - \xi^{l'(a+b+2)}F_2^{(l')}F_1^{(a+b+2)}\right)F_2^{(b+1)} \in \mathbf{u}^-.$$

Note that

$$F_1^{(a+b+2)}F_2^{(l'+b+1)}s = F_1^{(a+b+2)}F_2^{(l'+b+1)}F_1^{(l'-a-1)}\tilde{1}_{\lambda+(l'-a-1)\alpha_1} = \xi^{l'(b+1)}xs.$$

Using Subsection 1.1(c) we see that $x\tilde{1}_\lambda$ is a primitive element of weight $\lambda - (a + b + 2)\alpha_1 - (l' + b + 1)\alpha_2$. It is easy to see that $x\tilde{1}_\lambda$ is not a maximal element, namely $E_2 x\tilde{1}_\lambda \neq 0$.

Similarly we consider the homomorphism,

$$\tilde{Z}(\lambda) \to \tilde{Z}(\lambda + (l' - b - 1)\alpha_2)), \qquad \tilde{1}_\lambda \to t = F_2^{(l'-b-1)}\tilde{1}_{\lambda+(l'-b-1)\alpha_2}.$$

Let

$$y = \left(F_2^{(a+b+2)}F_1^{(l')} - \xi^{l'(a+b+2)}F_1^{(l')}F_2^{(a+b+2)}\right)F_1^{(a+1)} \in \mathbf{u}^-.$$

Note that

$$F_2^{(a+b+2)}F_1^{(l'+a+1)}t = F_2^{(a+b+2)}F_1^{(l'+a+1)}F_2^{(l'-b-1)}\tilde{1}_{\lambda+(l'-b-1)\alpha_2} = \xi^{l'(a+1)}yt.$$

Using Subsection 1.1(c) we see that $y\tilde{1}_\lambda$ is a primitive element of weight $\lambda - (l' + a + 1)\alpha_1 - (a + b + 2)\alpha_2$. It is easy to see that $y\tilde{1}_\lambda$ is not a maximal element.

Since $F_2^{(l'-a-b-2)}F_1^{(l'-b-1)} \in F_1^{(a+1)}\mathbf{u}^-$ (see Subsection 2.1(b)), we have a surjective homomorphism

$$\tilde{\mathbf{u}}F_1^{(a+1)}\tilde{1}_\lambda \to \tilde{\mathbf{u}}F_2^{(l'-a-b-2)}F_1^{(l'-b-1)}\tilde{1}_{\lambda+(l'-a-b-2)\alpha_1+(l'-a-b-2)\alpha_2},$$

$$F_1^{(a+1)}\tilde{1}_\lambda \to F_2^{(l'-a-b-2)}F_1^{(l'-b-1)}\tilde{1}_{\lambda+(l'-a-b-2)\alpha_1+(l'-a-b-2)\alpha_2}.$$

Let $z' = (F_1^{(l'-a-1)}F_2^{(l')} - \xi^{l'(l'-a-1)}F_2^{(l')}F_1^{(l'-a-1)}) \in \mathbf{u}^-$. Then

$$F_1^{(l'-a-1)}F_2^{(2l'-a-b-2)}F_1^{(l'-b-1)} = \xi^{l'(l'-a-b-2)}z'F_2^{(l'-a-b-2)}F_1^{(l'-b-1)}.$$

Let $z = z'F_1^{(a+1)} = F_1^{(l'-a-1)}F_2^{(l')}F_1^{(a+1)}$. According to Subsection 1.1(c) we see $z\tilde{1}_\lambda$ is a primitive element of weight $\lambda - l'\alpha_1 - l'\alpha_2$. We also can choose z to be $F_2^{(l'-b-1)}F_1^{(l')}F_2^{(b+1)}$ by consider the homomorphism

$$\tilde{\mathbf{u}}F_2^{(b+1)}\tilde{1}_\lambda \to \tilde{\mathbf{u}}F_1^{(l'-a-b-2)}F_2^{(l'-a-1)}\tilde{1}_{\lambda+(l'-a-b-2)\alpha_1+(l'-a-b-2)\alpha_2}.$$

Since all the weights of $x\tilde{1}_\lambda, y\tilde{1}_\lambda, z\tilde{1}_\lambda$ are smaller than $\lambda - (a + b + 2)\alpha_1 - (a + b + 2)\alpha_2$ and $F_2^{(a+1)}F_1^{(a+b+2)}F_2^{(b+1)}\tilde{1}_\lambda$ generates the unique irreducible submodule of $\tilde{Z}(\lambda)$, we see there are no maximal elements in $\tilde{Z}(\lambda)$ which have the same weight with $x\tilde{1}_\lambda$ or $y\tilde{1}_\lambda$ or $z\tilde{1}_\lambda$.

We have proved (ii).

According to Subsection 1.1(b) we see $\tilde{Z}(\lambda)$ has the nine composition factors. Note that the dimensions of irreducible $\tilde{\mathbf{u}}$-modules are known. Comparing the dimensions we know $\tilde{Z}(\lambda)$ has only the above composition factors. Thus (iii) is proved.

Part (iv) is due to [DS, I, K], which also can be checked directly by using (i)–(ii).

The theorem is proved.

THEOREM 2.3. *Let* $\lambda' = (a, b) \in \mathbf{Z}^2_{+,\mathbf{l}}$, $\lambda'' \in \mathbf{Z}^2$. *Set* $\lambda = \lambda' + \mathbf{l}\lambda'' \in \mathbf{Z}^2$. *Assume that* $0 < a + 1, b + 1 < l'$ *and* $a + b + 2 > l'$. *Then*

(i) *The following* 7 *elements are maximal in* $\tilde{Z}(\lambda)$,

$$\tilde{1}_\lambda, \qquad F_1^{(a+1)}\tilde{1}_\lambda, \qquad F_2^{(b+1)}\tilde{1}_\lambda, \qquad F_2^{(a+b+2-l')}F_1^{(a+1)}\tilde{1}_\lambda,$$
$$F_1^{(a+b+2-l')}F_2^{(b+1)}\tilde{1}_\lambda, \qquad F_2^{(a+1)}F_1^{(a+b+2)}F_2^{(b+1)}\tilde{1}_\lambda,$$
$$\sum_{0 \le r \le a+b+2-l'} \begin{bmatrix} b+1-r \\ l'-a-1 \end{bmatrix}_\xi^{-1} \xi^{(a+b+2-l'-r)(b+1-r)}$$
$$\times F_2^{(a+b+2-l'-r)} F_{12}'^{(r)} F_1^{(a+b+2-l'-r)}\tilde{1}_\lambda.$$

(ii) *The following* 2 *elements are primitive elements in* $\tilde{Z}(\lambda)$ *but not maximal*:

$$\left(F_1^{(a+1)}F_2^{(l')} - \xi^{l'(a+1)}F_2^{(l')}F_1^{(a+1)}\right)\tilde{1}_\lambda,$$
$$\left(F_2^{(b+1)}F_1^{(l')} - \xi^{l'(b+1)}F_1^{(l')}F_2^{(b+1)}\right)\tilde{1}_\lambda.$$

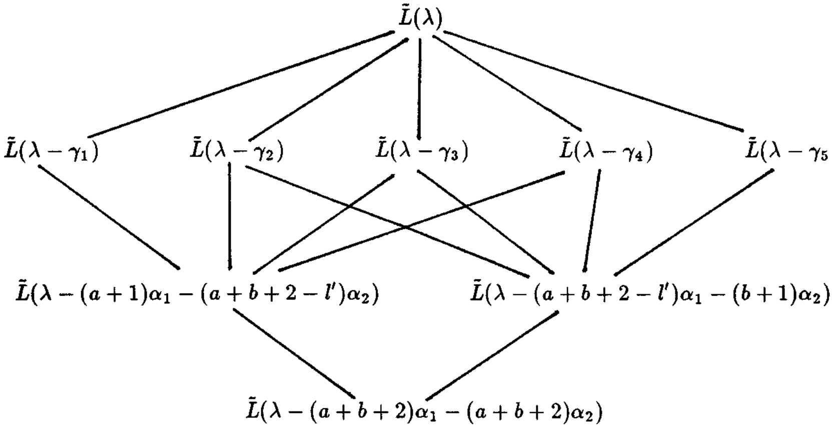
Moreover there are no maximal elements in $\tilde{Z}(\lambda)$ *which have the same weight with any of above two elements*.

(iii) *The maximal and primitive elements in* (i)–(ii) *provide nine composition factors of* $\tilde{Z}(\lambda)$ *which are*

$$\tilde{L}(\lambda), \qquad \tilde{L}(\lambda - (a+1)\alpha_1), \qquad \tilde{L}(\lambda - (b+1)\alpha_2),$$
$$\tilde{L}(\lambda - (a+1)\alpha_1 - (a+b+2-l')\alpha_2),$$
$$\tilde{L}(\lambda - (a+b+2-l')\alpha_1 - (b+1)\alpha_2)$$
$$\tilde{L}(\lambda - (a+b+2)\alpha_1 - (a+b+2)\alpha_2),$$
$$\tilde{L}(\lambda - (a+b+2-l')\alpha_1 - (a+b+2-l')\alpha_2)$$
$$\tilde{L}(\lambda - (a+1)\alpha_1 - l'\alpha_2), \qquad \tilde{L}(\lambda - l'\alpha_1 - (b+1)\alpha_2).$$

Moreover, $\tilde{Z}(\lambda)$ *has only nine composition factors.*

(iv) *The submodule lattice of* $\tilde{Z}(\lambda)$ *is* (*see* [DS, I, K])



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where $\gamma_1=(a+1)\alpha_1$, $\gamma_2=(a+1)\alpha_1+l'\alpha_2$, $\gamma_3=(a+b+2-l')\alpha_1+(a+b+2-l')\alpha_2$, $\gamma_4=l'\alpha_1+(b+1)\alpha_2$, $\gamma_5=(b+1)\alpha_2$.

Proof. According to Subsections 1.1(e)–(f) and 2.1(a) we see the first 6 elements in (i) are maximal.

Consider the homomorphism

$$\tilde{Z}(\lambda)\to\tilde{Z}(\lambda+(l'-a-1)\alpha_1)),\qquad \tilde{1}_\lambda\to s=F_1^{(l'-a-1)}\tilde{1}_{\lambda+(l'-a-1)\alpha_1}.$$

Let

$$x=\sum_{r=0}^{a+b+2-l'}\begin{bmatrix} b+1-r\\ l'-a-1\end{bmatrix}_\xi^{-1}\xi^{(a+b+2-l'-r)(b+1-r)}F_2^{(a+b+2-l'-r)}$$
$$\times F_{12}'^{(r)}F_1^{(a+b+2-l'-r)}\in\mathbf{u}^-.$$

Note that (see Subsection 2.1(b))

$$F_1^{(b+1)}F_2^{(a+b+2-l')}\tilde{1}_{\lambda+(l'-a-1)\alpha_1}=xF_1^{(l'-a-1)}\tilde{1}_{\lambda+(l'-a-1)\alpha_1}.$$

Using Subsection 1.1(c) we see that $x\tilde{1}_\lambda$ is a primitive element of weight $\lambda-(a+b+2-l')\alpha_1-(a+b+2-l')\alpha_2$. We will prove that $x\tilde{1}_\lambda$ is maximal after establishing (iv).

Now we consider the homomorphism

$$\tilde{Z}(\lambda) \to \tilde{Z}(\lambda + (2l' - a - b - 2)\alpha_1 + (l' - b - 1)\alpha_2),$$

$$\tilde{1}_\lambda \to t = F_2^{(l'-b-1)} F_1^{(2l'-a-b-2)} \tilde{1}_{\lambda + (2l'-a-b-2)\alpha_1 + (l'-b-1)\alpha_2}.$$

Let

$$y = \left(F_1^{(a+1)} F_2^{(l')} - \xi^{l'(a+1)} F_2^{(l')} F_1^{(a+1)} \right) \in \mathbf{u}^-.$$

Note that

$$yt = \xi^{l'(l'-b-1)} F_1^{(a+1)} F_2^{(2l'-b-1)} F_1^{(2l'-a-b-2)} \tilde{1}_{\lambda + (2l'-a-b-2)\alpha_1 + (l'-b-1)\alpha_2}$$

is a maximal element in the image of $\tilde{Z}(\lambda)$. Using Subsection 1.1(c) we see that $y\tilde{1}_\lambda$ is a primitive element of weight $\lambda - (a+1)\alpha_1 - l'\alpha_2$. We have $E_2 y\tilde{1}_\lambda \neq 0$.

Similarly, by considering the homomorphism

$$\tilde{Z}(\lambda) \to \tilde{Z}(\lambda + (l' - a - 1)\alpha_1 + (2l' - a - b - 2)\alpha_2),$$

$$\tilde{1}_\lambda \to t = F_1^{(l'-a-1)} F_2^{(2l'-a-b-2)} \tilde{1}_{\lambda + (l'-a-1)\alpha_1 + (2l'-a-b-2)\alpha_2},$$

we see that $z\tilde{1}_\lambda = (F_2^{(b+1)} F_1^{(l')} - \xi^{l'(b+1)} F_1^{(l')} F_2^{(b+1)})\tilde{1}_\lambda$ is a primitive element. We have $E_1 z\tilde{1}_\lambda \neq 0$. We will argue in the end of the proof for the second assertion of (ii).

The nine maximal and primitive elements provide nine composition factors listed in (iii). Comparing the dimensions we see $\tilde{Z}(\lambda)$ has only the nine composition factors.

By a direct computation we get (iv).

Now we can see easily that the 7th element in (i) is maximal. A simple computation shows that $E_i y\tilde{1}_\lambda$ and $E_i z\tilde{1}_\lambda$ $(i = 1, 2)$ are contained in the submodule of $\tilde{Z}(\lambda)$ generated by $F_2^{(a+b+2-l')} F_1^{(a+1)} \tilde{1}_\lambda$ and $F_1^{(a+b+2-l')} F_2^{(b+1)} \tilde{1}_\lambda$. Thus $\lambda - (a+b+2-l')\alpha_1 - (a+b+2-l')\alpha_2$ is a maximal weight of the maximal submodule of $\tilde{Z}(\lambda)$, so the 7th element in (i) is maximal.

Now we prove the second assertion of (ii). Assume m is a maximal element in $\tilde{Z}(\lambda)$ of weight $\tau = \lambda - (a+1)\alpha_1 - l'\alpha_2$. Since $\tilde{Z}(\lambda)$ has only one composition factor of highest weight τ, by Subsections 1.1(b) and 1.1(e), we see $m \in \tilde{\mathbf{u}} y\tilde{1}_\lambda$. Let $m_1 = F_2^{(a+b+2-l')} F_1^{(a+1)} \tilde{1}_\lambda$ and $m_2 = F_1^{(a+b+2-l')} F_2^{(b+1)} \tilde{1}_\lambda$. By (iv) we see $m \notin \tilde{\mathbf{u}} m_1 + \tilde{\mathbf{u}} m_2$. It is easy to check that the τ-weight space of $\tilde{\mathbf{u}} m_1$ is zero. Thus we have $m = hy\tilde{1}_\lambda + r$ for some $r \in \tilde{\mathbf{u}} m_2$ and some nonzero $h \in \mathbf{Q}(\xi)$. Applying $E_2^{(2l'-a-b-2)}$ to both sides of $m = hy\tilde{1}_\lambda + r$ we see that $m_1 \in \tilde{\mathbf{u}} m_2$, this is not true. Similarly we see there are no maximal elements in $Z(\lambda)$ of weight $\lambda - l'\alpha_1 - (b+1)\alpha_2$.

The theorem is proved.

Completely as the arguments for Theorems 2.2–2.3 we get the following results.

THEOREM 2.4. *Let* $\lambda' = (a, b) \in \mathbf{Z}^2_{+,\mathbf{l}}$, $\lambda'' \in \mathbf{Z}^2$. *Set* $\lambda = \lambda' + \mathbf{l}\lambda'' \in \mathbf{Z}^2$. *Assume that* $0 < a + 1, b + 1 < l'$ *and* $a + b + 2 = l'$. *Then*

(i) *The following* 4 *elements are maximal in* $\tilde{Z}(\lambda)$:

$$\tilde{1}_\lambda, \qquad F_1^{(a+1)}\tilde{1}_\lambda, \qquad F_2^{(b+1)}\tilde{1}_\lambda, \qquad F_2^{(a+1)}F_1^{(l')}F_2^{(b+1)}\tilde{1}_\lambda.$$

(ii) *The maximal elements in* (i) *provide four composition factors of* $\tilde{Z}(\lambda)$, *which are* $\tilde{L}(\lambda)$, $\tilde{L}(\lambda - (a+1)\alpha_1)$, $\tilde{L}(\lambda - (b+1)\alpha_2)$, $\tilde{L}(\lambda - l'\alpha_1 - l'\alpha_2)$. *Moreover*, $\tilde{Z}(\lambda)$ *has only the four composition factors*.

(iii) *The submodule lattice of* $\tilde{Z}(\lambda)$ *is as follows* (*see* [DS, I, K]),

$$\begin{tikzcd}[arrows=dash]
 & \tilde{L}(\lambda) \arrow[dl] \arrow[dr] & \\
\tilde{L}(\lambda - (a+1)\alpha_1) \arrow[dr] & & \tilde{L}(\lambda - (b+1)\alpha_2) \arrow[dl] \\
 & \tilde{L}(\lambda - l'\alpha_1 - l'\alpha_2) &
\end{tikzcd}$$

THEOREM 2.5. *Let* $\lambda' = (l' - 1, b) \in \mathbf{Z}^2_{+,\mathbf{l}}$, $\lambda'' \in \mathbf{Z}^2$. *Set* $\lambda = \lambda' + \mathbf{l}\lambda'' \in \mathbf{Z}^2$. *Assume that* $0 < b + 1 < l'$. *Then*

(i) *The following* 3 *elements are maximal in* $\tilde{Z}(\lambda)$:

$$\tilde{1}_\lambda, \qquad F_2^{(b+1)}\tilde{1}_\lambda, \qquad F_1^{(b+1)}F_2^{(b+1)}\tilde{1}_\lambda.$$

(ii) *The element* $x = (F_2^{(b+1)}F_1^{(l')} - \xi^{l'(b+1)}F_1^{(l')}F_2^{(b+1)})\tilde{1}_\lambda$ *is primitive. There are no maximal elements in* $\tilde{Z}(\lambda)$ *of weight* $\lambda - l'\alpha_1 - (b+1)\alpha_2$.

(iii) *The maximal and primitive elements in* (i)–(ii) *provide four composition factors of* $\tilde{Z}(\lambda)$ *which are* $\tilde{L}(\lambda)$, $\tilde{L}(\lambda - (b + 1 - \alpha_2)$, $\tilde{L}(\lambda - (b+1)\alpha_1 - (b+1)\alpha_2)$, $\tilde{L}(\lambda - l'\alpha_1 - (b+1)\alpha_2)$. *Moreover*, $\tilde{Z}(\lambda)$ *has only the four composition factors*.

(iv) *The submodule lattice of* $\tilde{Z}(\lambda)$ *is as follows* (*see* [DS, I, K]),

$$\begin{tikzcd}[column sep=tiny] & \tilde{L}(\lambda) \arrow[dl, dash] \arrow[dr, dash] & \\ \tilde{L}(\lambda - l'\alpha_1 - (b+1)\alpha_2) \arrow[dr, dash] & & \tilde{L}(\lambda - (b+1)\alpha_2) \arrow[dl, dash] \\ & \tilde{L}(\lambda - (b+1)\alpha_1 - (b+1)\alpha_2) & \end{tikzcd}$$

THEOREM 2.6. *Let* $\lambda' = (a, l' - 1) \in \mathbf{Z}^2_{+,\mathbf{l}}$, $\lambda'' \in \mathbf{Z}^2$. *Set* $\lambda = \lambda' + \mathbf{l}\lambda'' \in \mathbf{Z}^2$. *Assume that* $0 < a + 1 < l'$. *Then*

(i) *The following* 3 *elements are maximal in* $\tilde{Z}(\lambda)$:

$$\tilde{1}_\lambda, \qquad F_1^{(a+1)}\tilde{1}_\lambda, \qquad F_2^{(a+1)}F_1^{(a+1)}\tilde{1}_\lambda.$$

(ii) *The element* $x = (F_1^{(a+1)}F_2^{(l')} - \xi^{l'(a+1)}F_2^{(l')}F_1^{(a+1)})\tilde{1}_\lambda$ *is primitive. There are no maximal elements in* $\tilde{Z}(\lambda)$ *of weight* $\lambda - (a+1)\alpha_1 - l'\alpha_2$.

(iii) *The maximal and primitive elements in* (i)–(ii) *provide four composition factors of* $\tilde{Z}(\lambda)$, *which are* $\tilde{L}(\lambda)$, $\tilde{L}(\lambda - (a+1)\alpha_1)$, $\tilde{L}(\lambda - (a+1)\alpha_1 - (a+1)\alpha_2)$, $\tilde{L}(\lambda - (a+1)\alpha_1 - l'\alpha_2)$. *Moreover*, $\tilde{Z}(\lambda)$ *has only the four composition factors*.

(iv) *The submodule lattice of* $\tilde{Z}(\lambda)$ *is as follows* (*see* [DS, I, K]):

$$\begin{tikzcd}[column sep=tiny] & \tilde{L}(\lambda) \arrow[dl, dash] \arrow[dr, dash] & \\ \tilde{L}(\lambda - (b+1)\alpha_1) \arrow[dr, dash] & & \tilde{L}(\lambda - (a+1)\alpha_1 - l'\alpha_2) \arrow[dl, dash] \\ & \tilde{L}(\lambda - (a+1)\alpha_1 - (a+1)\alpha_2) & \end{tikzcd}$$

THEOREM 2.7. *Let* $\lambda' = (l' - 1, l' - 1) \in \mathbf{Z}^2_{+,\mathbf{l}}$, $\lambda'' \in \mathbf{Z}^2$. *Set* $\lambda = \lambda' + \mathbf{l}\lambda'' \in \mathbf{Z}^2$. *Then* $\tilde{Z}(\lambda)$ *is irreducible*. (*see* [X1]).

3. STRUCTURE OF WEYL MODULES FOR TYPE A_2

For $\lambda = (\lambda_1, \lambda_2) \in \mathbf{Z}^2_+$ we denote by I_λ the left ideal of U_ξ generated by all $E_i^{(a)}$ $(a \geq 1)$, $F_i^{(a_i)}$ $(a_i \geq \lambda_i + 1), K_i - \xi^{\lambda_i}, \left[{K_i, c \atop a}\right] - \left[{\lambda_i + c \atop a}\right]_\xi$. The Weyl module $V(\lambda)$ of U_ξ is defined to be U_ξ/I_λ; its dimension is $(\lambda_1+1)(\lambda_2+1)(\lambda_1+\lambda_2+2)/2$. Let v_λ be a nonzero element in $V(\lambda)_\lambda$. A similar discussion as in Section 2 leads to the following results.

THEOREM 3.1. *Let $\lambda'=(a,b)\in \mathbf{Z}^2_{+,\mathbf{l}}$, $\lambda''=(c,d)\in \mathbf{Z}^2_+$. Set $\lambda=\lambda'+\mathbf{l}\lambda''\in \mathbf{Z}^2_+$. Assume that $a+b+2<l'$ and $1\le c,d$. Then*

(i) *The following* 6 *elements are maximal in $V(\lambda)$*:

$$v_\lambda,\qquad F_1^{(a+1)}v_\lambda,\qquad F_2^{(b+1)}v_\lambda,\qquad F_2^{(a+b+2)}F_1^{(a+1)}v_\lambda,$$
$$F_1^{(a+b+2)}F_2^{(b+1)}v_\lambda,\qquad F_2^{(a+1)}F_1^{(a+b+2)}F_2^{(b+1)}v_\lambda.$$

(ii) *The following* 3 *elements are primitive elements in $V(\lambda)$ but not maximal*:

$$\left(dF_1^{(a+b+2)}F_2^{(l')}-(d-1)\xi^{l'(a+b+2)}F_2^{(l')}F_1^{(a+b+2)}\right)F_2^{(b+1)}v_\lambda \qquad (d\ge 2)$$
$$\left(cF_2^{(a+b+2)}F_1^{(l')}-(c-1)\xi^{l'(a+b+2)}F_1^{(l')}F_2^{(a+b+2)}\right)F_1^{(a+1)}v_\lambda \qquad (c\ge 2)$$
$$F_1^{(l'-a-1)}F_2^{(l')}F_1^{(a+1)}v_\lambda.$$

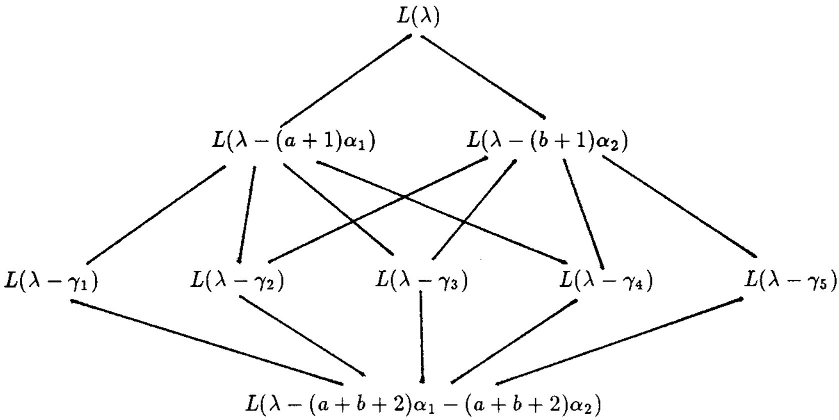
(We also can choose the third one to be $F_2^{(l'-b-1)}F_1^{(l')}F_2^{(b+1)}v_\lambda$.) Moreover there are no maximal elements in $V(\lambda)$ which have the same weight with any of the above three elements.

(iii) *The maximal and primitive elements in* (i)–(ii) *provide the following composition factors of $V(\lambda)$,*

$$L(\lambda),\qquad L(\lambda-(a+1)\alpha_1),\qquad L(\lambda-(b+1)\alpha_2),$$
$$L(\lambda-(a+1)\alpha_1-(a+b+2)\alpha_2),$$
$$L(\lambda-(a+b+2)\alpha_1-(b+1)\alpha_2)$$
$$L(\lambda-(a+b+2)\alpha_1-(a+b+2)\alpha_2),$$
$$L(\lambda-(a+b+2)\alpha_1-(l'+b+1)\alpha_2)\ (d\ge 2),$$
$$L(\lambda-(l'+a+1)\alpha_1-(a+b+2)\alpha_2)\ (c\ge 2),$$
$$L(\lambda-l'\alpha_1-l'\alpha_2).$$

Moreover, $V(\lambda)$ has only the above composition factors.

(iv) *If* $c, d \geq 2$, *the submodule lattice of* $V(\lambda)$ *is as follows* (*see* [DS, I, K]),



where $\gamma_1 = (l' + a + 1)\alpha_1 + (a + b + 2)\alpha_2$, $\gamma_2 = (a + 1)\alpha_1 + (a + b + 2)\alpha_2$, $\gamma_3 = l'\alpha_1 + l'\alpha_2$, $\gamma_4 = (a + b + 2)\alpha_1 + (b + 1)\alpha_2$, $\gamma_5 = (a + b + 2)\alpha_1 + (l' + b + 1)\alpha_2$. *When* $c = 1, d \geq 2$ (*resp.* $c \geq 2, d = 1$; *or* $c = d = 1$), *the submodule lattice of* $V(\lambda)$ *is obtained from the above lattice by deleting* $L(\lambda - \gamma_1)$ *and the two segments connecting* $L(\lambda - \gamma_1)$ (*resp.* $L(\lambda - \gamma_5)$ *and the two segments connecting* $L(\lambda - \gamma_5)$; *or* $L(\lambda - \gamma_1)$, $L(\lambda - \gamma_5)$ *and the four segments connecting any of the two modules*).

THEOREM 3.2. *Let* $\lambda' = (a, b) \in \mathbf{Z}^2_{+,\mathbf{l}}$, $\lambda'' = (c, d) \in \mathbf{Z}^2_+$. *Set* $\lambda = \lambda' + \mathbf{l}\lambda'' \in \mathbf{Z}^2_+$. *Assume that* $0 < a + 1, b + 1 < l', a + b + 2 > l'$, *and* $1 \leq c, d$. *Then*

(i) *The following* 7 *elements are maximal in* $V(\lambda)$:

$$v_\lambda, \qquad F_1^{(a+1)}v_\lambda, \qquad F_2^{(b+1)}v_\lambda, \qquad F_2^{(a+b+2-l')}F_1^{(a+1)}v_\lambda,$$
$$F_1^{(a+b+2-l')}F_2^{(b+1)}v_\lambda, \qquad F_2^{(a+1)}F_1^{(a+b+2)}F_2^{(b+1)}v_\lambda,$$
$$\sum_{0 \leq r \leq a+b+2-l'} \begin{bmatrix} b+1-r \\ l'-a-1 \end{bmatrix}_\xi^{-1}$$
$$\times\, \xi^{(a+b+2-l'-r)(b+1-r)} F_2^{(a+b+2-l'-r)} F_{12}'^{(r)} F_1^{(a+b+2-l'-r)} v_\lambda.$$

(ii) *The following* 2 *elements are primitive elements in* $V(\lambda)$ *but not maximal*:

$$\left((d+1)F_1^{(a+1)}F_2^{(l')} - d\xi^{l'(a+1)}F_2^{(l')}F_1^{(a+1)}\right)v_\lambda,$$
$$\left((c+1)F_2^{(b+1)}F_1^{(l')} - c\xi^{l'(b+1)}F_1^{(l')}F_2^{(b+1)}\right)v_\lambda.$$

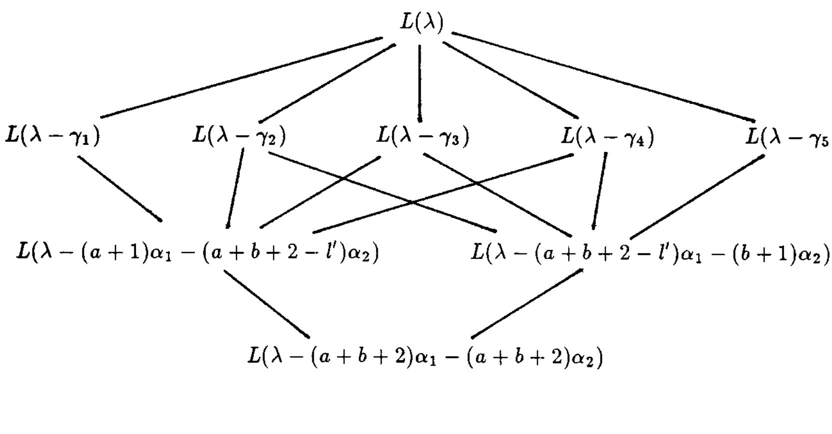
Moreover there are no maximal elements in $V(\lambda)$ *which have the same weight with any of above two elements.*

(iii) *The maximal and primitive elements in* (i)–(ii) *provide nine composition factors of* $V(\lambda)$, *which are*

$$L(\lambda), \qquad L(\lambda-(a+1)\alpha_1), \qquad L(\lambda-(b+1)\alpha_2),$$
$$L(\lambda-(a+1)\alpha_1-(a+b+2-l')\alpha_2),$$
$$L(\lambda-(a+b+2-l')\alpha_1-(b+1)\alpha_2)$$
$$L(\lambda-(a+b+2)\alpha_1-(a+b+2)\alpha_2),$$
$$L(\lambda-(a+b+2-l')\alpha_1-(a+b+2-l')\alpha_2),$$
$$L(\lambda-(a+1)\alpha_1-l'\alpha_2), \qquad L(\lambda-l'\alpha_1-(b+1)\alpha_2).$$

Moreover, $V(\lambda)$ *has only the nine composition factors.*

(iv) *The submodule lattice of* $V(\lambda)$ *is* (*see* [DS, I, K])



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where $\gamma=(a+1)\alpha_1$, $\gamma_2=(a+1)\alpha_1+l'\alpha_2$, $\gamma_3=(a+b+2-l')\alpha_1+(a+b+2-l')\alpha_2$, $\gamma_4=l'\alpha_1+(b+1)\alpha_2$, $\gamma_5=(b+1)\alpha_2$.

THEOREM 3.3. *Let* $\lambda' = (a, b) \in \mathbf{Z}^2_{+,\mathbf{l}}$, $\lambda'' = (c, d) \in \mathbf{Z}^2_{+}$. *Set* $\lambda = \lambda' + \mathbf{l}\lambda'' \in \mathbf{Z}^2_{+}$. *Assume that* $0 < a + 1, b + 1 < l'$, $a + b + 2 = l'$ *and* $1 \leq c, d$. *Then*

(i) *The following* 4 *elements are maximal in* $V(\lambda)$:

$$v_\lambda, \qquad F_1^{(a+1)} v_\lambda, \qquad F_2^{(b+1)} v_\lambda, \qquad F_2^{(a+1)} F_1^{(l')} F_2^{(b+1)} v_\lambda.$$

(ii) *The maximal elements in* (i) *provide four composition factors of* $V(\lambda)$, *which are* $L(\lambda)$, $L(\lambda - (a+1)\alpha_1)$, $L(\lambda - (b+1)\alpha_2)$, $L(\lambda - l'\alpha_1 - l'\alpha_2)$. *Moreover*, $V(\lambda)$ *has only the four composition factors*.

(iii) *The submodule lattice of* $V(\lambda)$ *is as follows* (*see* [DS, I, K]),

$$\begin{tikzcd}[column sep=small]
 & L(\lambda) \arrow[dl, dash] \arrow[dr, dash] & \\
L(\lambda - (a+1)\alpha_1) \arrow[dr, dash] & & L(\lambda - (b+1)\alpha_2) \arrow[dl, dash] \\
 & L(\lambda - l'\alpha_1 - l'\alpha_2) &
\end{tikzcd}$$

THEOREM 3.4. *Let* $\lambda' = (l' - 1, b) \in \mathbf{Z}^2_{+,\mathbf{l}}$, $\lambda'' = (c, d) \in \mathbf{Z}^2_{+}$. *Set* $\lambda = \lambda' + \mathbf{l}\lambda'' \in \mathbf{Z}^2_{+}$. *Assume that* $0 < b + 1 < l'$ *and* $1 \leq c, d$. *Then*

(i) *The following* 3 *elements are maximal in* $V(\lambda)$:

$$v_\lambda, \qquad F_2^{(b+1)} v_\lambda, \qquad F_1^{(b+1)} F_2^{(b+1)} v_\lambda.$$

(ii) *The element* $x = ((c+1) F_2^{(b+1)} F_1^{(l')} - c\xi^{l'(b+1)} F_1^{(l')} F_2^{(b+1)}) v_\lambda$ *is primitive. There are no maximal elements in* $V(\lambda)$ *of weight* $\lambda - l'\alpha_1 - (b + 1)\alpha_2$.

(iii) *The maximal and primitive elements in* (i)–(ii) *provide four composition factors of* $V(\lambda)$, *which are* $L(\lambda)$, $L(\lambda - (b+1)\alpha_2)$, $L(\lambda - (b+1)\alpha_1 - (b+1)\alpha_2)$, $L(\lambda - l'\alpha_1 - (b+1)\alpha_2)$. *Moreover*, $V(\lambda)$ *has only the four composition factors*.

(iv) *The submodule lattice of* $V(\lambda)$ *is as follows* (*see* [DS, I, K]),

$$\begin{tikzcd}[column sep=tiny] & L(\lambda) \arrow[dl, dash] \arrow[dr, dash] & \\ L(\lambda - l'\alpha_1 - (b+1)\alpha_2) \arrow[dr, dash] & & L(\lambda - (b+1)\alpha_2) \arrow[dl, dash] \\ & L(\lambda - (b+1)\alpha_1 - (b+1)\alpha_2) & \end{tikzcd}$$

THEOREM 3.5. *Let* $\lambda' = (a, l'-1) \in \mathbf{Z}^2_{+,\mathbf{l}}$, $\lambda'' = (c, d) \in \mathbf{Z}^2_+$. *Set* $\lambda = \lambda' + \mathbf{l}\lambda'' \in \mathbf{Z}^2_+$. *Assume that* $0 < a+1 < l'$ *and* $1 \le c, d$. *Then*

(i) *The following* 3 *elements are maximal in* $V(\lambda)$:

$$v_\lambda, \qquad F_1^{(a+1)} v_\lambda, \qquad F_2^{(a+1)} F_1^{(a+1)} v_\lambda.$$

(ii) *The element* $x = ((d+1)F_1^{(a+1)}F_2^{(l')} - d\xi^{l'(a+1)}F_2^{(l')}F_1^{(a+1)})v_\lambda$ *is primitive. There are no maximal elements in* $V(\lambda)$ *of weight* $\lambda - (a+1)\alpha_1 - l'\alpha_2$.

(iii) *The maximal and primitive elements in* (i)–(ii) *provide four composition factors of* $V(\lambda)$, *which are* $L(\lambda)$, $L(\lambda - (a+1)\alpha_1)$, $L(\lambda - (a+1)\alpha_1 - (a+1)\alpha_2)$, $L(\lambda - (a+1)\alpha_1 - l'\alpha_2)$. *Moreover,* $V(\lambda)$ *has only the four composition factors.*

(iv) *The submodule lattice of* $V(\lambda)$ *is as follows* (*see* [DS, I, K]),

$$\begin{tikzcd}[column sep=tiny] & L(\lambda) \arrow[dl, dash] \arrow[dr, dash] & \\ L(\lambda - (a+1)\alpha_1) \arrow[dr, dash] & & L(\lambda - (a+1)\alpha_1 - l'\alpha_2) \arrow[dl, dash] \\ & L(\lambda - (a+1)\alpha_1 - (a+1)\alpha_2) & \end{tikzcd}$$

THEOREM 3.6. *Let* $\lambda' = (l'-1, l'-1) \in \mathbf{Z}^2_{+,\mathbf{l}}$, $\lambda'' \in \mathbf{Z}^2_+$. *Set* $\lambda = \lambda' + \mathbf{l}\lambda'' \in \mathbf{Z}^2_+$. *Then* $V(\lambda)$ *is irreducible* (*see* [X1].)

THEOREM 3.7. *Let* $\lambda' = (a, b) \in \mathbf{Z}^2_{+,\mathbf{l}}$, $\lambda'' = (c, 0) \in \mathbf{Z}^2_+$. *Set* $\lambda = \lambda' + \mathbf{l}\lambda'' \in \mathbf{Z}^2_+$. *Assume that* $a + b + 2 < l'$ *and* $1 \le c$. *Then*

(i) *The following* v_λ, $F_1^{(a+1)}v_\lambda$, $F_2^{(a+b+2)}F_1^{(l'+a+1)}v_\lambda$ $(c \ge 2)$ *are maximal in* $V(\lambda)$.

(ii) *The maximal elements in* (i) *provide the following composition factors of* $V(\lambda)$: $L(\lambda)$, $L(\lambda - (a+1)\alpha_1)$, $L(\lambda - (l' + a + 1)\alpha_1 - (a + b$

$+ 2)\alpha_2)$ $(c \geq 2)$. *Moreover, $V(\lambda)$ has only these composition factors. The submodule lattice of $V(\lambda)$ is trivial.*

THEOREM 3.7′. *Let $\lambda' = (a, b) \in \mathbf{Z}^2_{+,\mathbf{l}}$, $\lambda'' = (0, d) \in \mathbf{Z}^2_+$. Set $\lambda = \lambda' + \mathbf{l}\lambda'' \in \mathbf{Z}^2_+$. Assume that $a + b + 2 < l'$ and $1 \leq d$. Then*

(i) *The following v_λ, $F_2^{(b+1)}v_\lambda$, $F_1^{(a+b+2)}F_2^{(l'+b+1)}v_\lambda (d \geq 2)$ are maximal in $V(\lambda)$.*

(ii) *The maximal elements in* (i) *provide the following composition factors of $V(\lambda)$, $L(\lambda)$, $L(\lambda - (b + 1)\alpha_2)$, $L(\lambda - (a + b + 2)\alpha_1 - (l' + b + 1)\alpha_2)$ $(d \geq 2)$. Moreover, $V(\lambda)$ has only these composition factors. The submodule lattice of $V(\lambda)$ is trivial.*

THEOREM 3.8. *Let $\lambda' = (a, b) \in \mathbf{Z}^2_{+,\mathbf{l}}$, $\lambda'' = (c, 0) \in \mathbf{Z}^2_+$. Set $\lambda = \lambda' + \mathbf{l}\lambda'' \in \mathbf{Z}^2_+$. Assume that $0 < a + 1, b + 1 < l', a + b + 2 > l'$, and $1 \leq c$. Then*

(i) *The following 4 elements are maximal in $V(\lambda)$,*

$$v_\lambda, \qquad F_1^{(a+1)}v_\lambda, \qquad F_2^{(a+b+2-l')}F_1^{(a+1)}v_\lambda,$$

$$\sum_{0 \leq r \leq a+b+2-l'} \begin{bmatrix} b+1-r \\ l'-a-1 \end{bmatrix}_\xi^{-1} \times \xi^{(a+b+2-l'-r)(b+1-r)} F_2^{(a+b+2-l'-r)} F_{12}'^{(r)} F_1^{(a+b+2-l'-r)} v_\lambda.$$

(ii) *The element $F_2^{(b+1)}F_1^{(l')}v_\lambda$ is a primitive element in $V(\lambda)$ but not maximal. Moreover there are no maximal elements in $V(\lambda)$ of weight $\lambda - l'\alpha_1 - (b + 1)\alpha_2$.*

(iii) *The maximal and positive elements in* (i)–(ii) *provide five composition factors of $V(\lambda)$, which are*

$$L(\lambda), \qquad L(\lambda - (a+1)\alpha_1),$$
$$L(\lambda - (a+1)\alpha_1 - (a+b+2-l')\alpha_2),$$
$$L(\lambda - (a+b+2-l')\alpha_1 - (a+b+2-l')\alpha_2),$$
$$L(\lambda - l'\alpha_1 - (b+1)\alpha_2).$$

Moreover, $V(\lambda)$ has only the five composition factors.

(iv) *The submodule lattice of $V(\lambda)$ is as follows (see* [DS, I, K]),

$$\begin{tikzcd}[arrows=dash] & L(\lambda) \arrow[dl] \arrow[d] \arrow[dr] & \\ L(\lambda - \gamma_2) \arrow[dr] & L(\lambda - \gamma_3) \arrow[d] & L(\lambda - \gamma_4) \arrow[dl] \\ & L(\lambda - (a+1)\alpha_1 - (a+b+2-l')\alpha_2) & \end{tikzcd}$$

where $\gamma_2 = (a+1)\alpha_1$, $\gamma_3 = (a+b+2-l')\alpha_1 + (a+b+2-l')\alpha_2$, $\gamma_4 = l'\alpha_1 + (b+1)\alpha_2$.

THEOREM 3.8′. *Let* $\lambda' = (a,b) \in \mathbf{Z}^2_{+,\mathbf{l}}$, $\lambda'' = (0,d) \in \mathbf{Z}^2_+$. *Set* $\lambda = \lambda' + \mathbf{l}\lambda'' \in \mathbf{Z}^2_+$. *Assume that* $0 < a+1, b+1 < l', a+b+2 > l'$, *and* $1 \le d$. *Then*

(i) *The following* 4 *elements are maximal in* $V(\lambda)$,

$$v_\lambda, \qquad F_2^{(b+1)}v_\lambda, \qquad F_1^{(a+b+2-l')}F_2^{(b+1)}v_\lambda,$$

$$\sum_{0\le r\le a+b+2-l'} \begin{bmatrix} b+1-r \\ l'-a-1 \end{bmatrix}_\xi^{-1} \times \xi^{(a+b+2-l'-r)(b+1-r)} F_2^{(a+b+2-l'-r)} F_{12}'^{(r)} F_1^{(a+b+2-l'-r)} v_\lambda.$$

(ii) *The element* $F_1^{(a+1)}F_2^{(l')}v_\lambda$ *is a primitive element in* $V(\lambda)$ *but not maximal. Moreover there are no maximal elements in* $V(\lambda)$ *of weight* $\lambda - (a+1)\alpha_1 - l'\alpha_2$.

(iii) *The maximal and primitive elements in* (i)–(ii) *provide five composition factors of* $V(\lambda)$, *which are*

$$L(\lambda), \qquad L(\lambda - (b+1)\alpha_2),$$
$$L(\lambda - (a+b+2-l')\alpha_1 - (b+1)\alpha_2),$$
$$L(\lambda - (a+b+2-l')\alpha_1 - (a+b+2-l')\alpha_2),$$
$$L(\lambda - (a+1)\alpha_1 - l'\alpha_2).$$

Moreover, $V(\lambda)$ *has only the five composition factors*.

(iv) *The submodule lattice of* $V(\lambda)$ *is as follows* (*see* [DS, I, K]),

$$\begin{tikzcd} & L(\lambda) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ L(\lambda-\gamma_2) \arrow[dr, dash] & L(\lambda-\gamma_3) \arrow[d, dash] & L(\lambda-\gamma_4) \arrow[dl, dash] \\ & L(\lambda-(a+b+2-l')\alpha_1-(b+1)\alpha_2) & \end{tikzcd}$$

where $\gamma_2 = (a+1)\alpha_1 + l'\alpha_2$, $\gamma_3 = (a+b+2-l')\alpha_1 + (a+b+2-l')\alpha_2$, $\gamma_4 = (b+1)\alpha_2$.

THEOREM 3.9. *Let* $\lambda' = (a,b) \in \mathbf{Z}^2_{+,\mathbf{l}}$, $\lambda'' = (c,0) \in \mathbf{Z}^2_+$. *Set* $\lambda = \lambda' + \mathbf{l}\lambda'' \in \mathbf{Z}^2_+$. *Assume that* $0 < a+1, b+1 < l', a+b+2 = l'$, *and* $l \le c$, *Then*

(i) *The two elements* v_λ, $F_1^{(a+1)}v_\lambda$ *are maximal in* $V(\lambda)$.

(ii) *The maximal elements in* (i) *provide two composition factors of* $V(\lambda)$, *which are* $L(\lambda)$, $L(\lambda - (a+1)\alpha_1)$. *Moreover*, $V(\lambda)$ *has only the two composition factors and the submodule lattice of* $V(\lambda)$ *is trivial*.

THEOREM 3.9′. *Let* $\lambda' = (a, b) \in \mathbf{Z}^2_{+,\mathbf{l}}$, $\lambda'' = (0, d) \in \mathbf{Z}^2_+$. *Set* $\lambda = \lambda' + \mathbf{l}\lambda'' \in \mathbf{Z}^2_+$. *Assume that* $0 < a + 1, b + 1 < l', a + b + 2 = l'$, *and* $1 \leq d$. *Then*

(i) *The two elements* v_λ, $F_2^{(b+1)} v_\lambda$ *are maximal in* $V(\lambda)$.

(ii) *The maximal elements in* (i) *provide two composition factors of* $V(\lambda)$, *which are* $L(\lambda)$, $L(\lambda - (b + 1)\alpha_2)$. *Moreover,* $V(\lambda)$ *has only the two composition factors and the submodule lattice of* $V(\lambda)$ *is trivial.*

THEOREM 3.10. *Let* $\lambda' = (l' - 1, b) \in \mathbf{Z}^2_{+,\mathbf{l}}$, $\lambda'' = (c, 0) \in \mathbf{Z}^2_+$. *Set* $\lambda = \lambda' + \mathbf{l}\lambda'' \in \mathbf{Z}^2_+$. *Assume that* $0 < b + 1 < l'$ *and* $1 \leq c$. *Then*

(i) *The two elements* v_λ, $F_2^{(b+1)} F_1^{(l')} v_\lambda$ *are maximal in* $V(\lambda)$.

(ii) *The maximal elements in* (i) *provide two composition factors of* $V(\lambda)$, *which are* $L(\lambda)$, $L(\lambda - l'\alpha_1 - (b + 1)\alpha_2)$. *Moreover,* $V(\lambda)$ *has only the two composition factors and the submodule lattice of* $V(\lambda)$ *is trivial.*

THEOREM 3.10′. *Let* $\lambda' = (l' - 1, b) \in \mathbf{Z}^2_{+,\mathbf{l}}$, $\lambda'' = (0, d) \in \mathbf{Z}^2_+$. *Set* $\lambda = \lambda' + \mathbf{l}\lambda'' \in \mathbf{Z}^2_+$. *Assume that* $0 < b + 1 < l'$ *and* $1 \leq d$. *Then*

(i) *The three elements* v_λ, $F_2^{(b+1)} v_\lambda$, $F_1^{(b+1)} F_2^{(b+1)} v_\lambda$ *are maximal in* $V(\lambda)$.

(ii) *The maximal elements in* (i) *provide three composition factors of* $V(\lambda)$, *which are* $L(\lambda)$, $L(\lambda - (b + 1)\alpha_2)$, $L(\lambda - (a + 1)\alpha_1 - (b + 1)\alpha_2)$. *Moreover,* $V(\lambda)$ *has only the three composition factors and the submodule lattice of* $V(\lambda)$ *is trivial.*

THEOREM 3.11. *Let* $\lambda' = (a, l' - 1) \in \mathbf{Z}^2_{+,\mathbf{l}}$, $\lambda'' = (c, 0) \in \mathbf{Z}^2_+$. *Set* $\lambda = \lambda' + \mathbf{l}\lambda'' \in \mathbf{Z}^2_+$. *Assume that* $0 < a + 1 < l'$ *and* $1 \leq c$. *Then*

(i) *The following* 3 *elements are maximal in* $V(\lambda)$:

$$v_\lambda, \qquad F_1^{(a+1)} v_\lambda, \qquad F_2^{(a+1)} F_1^{(a+1)} v_\lambda.$$

(ii) *The maximal elements in* (i) *provide three composition factors of* $V(\lambda)$, *which are* $L(\lambda)$, $L(\lambda - (a + 1)\alpha_1)$, $L(\lambda - (a + 1)\alpha_1 - (a + 1)\alpha_2)$. *Moreover,* $V(\lambda)$ *has only the three composition factors and the module structure is trivial.*

THEOREM 3.11′. *Let* $\lambda' = (a, l' - 1) \in \mathbf{Z}^2_{+,\mathbf{l}}$, $\lambda'' = (0, d) \in \mathbf{Z}^2_+$. *Set* $\lambda = \lambda' + \mathbf{l}\lambda'' \in \mathbf{Z}^2_+$. *Assume that* $0 < a + 1 < l'$ *and* $1 \leq d$. *Then*

(i) *The two elements* v_λ, $F_1^{(a+1)} F_2^{(l')} v_\lambda$ *are maximal in* $V(\lambda)$.

(ii) *The maximal elements in* (i) *provide two composition factors of* $V(\lambda)$, *which are* $L(\lambda)$, $L(\lambda - (a + 1)\alpha_1 - l'\alpha_2)$. *Moreover,* $V(\lambda)$ *has only the two composition factors and the submodule lattice is trivial.*

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